Deadline : 2025/11/28, 9:00.

1. Suppose that $g : \mathbb{R} \to [a, b]$ is a continuous function and f is integrable on $[a, b]$. Prove that

$$F(x) := \int_a^{g(x)} f(t)\, dt$$

is continuous on $\mathbb{R}$.

(**Hint**: Express $F(x)$ as a composite function of $g(x)$ and $G(x) := \int_a^x f(t)\, dt$.)

2. Evaluate the following limit.

   (a) $\displaystyle\lim_{n\to\infty} \frac{1}{n^{16}} \sum_{i=1}^{n} i^{15}$

   (b) $\displaystyle\lim_{n\to\infty} n^{-\frac{3}{2}} \sum_{i=1}^{n} \sqrt{i}$

   (c) $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{2n} \frac{i^2}{n^3}$

   (d) $\displaystyle\lim_{n\to\infty} \sum_{i=n}^{2n} \frac{1}{i}$

3. **The error function** The *error function*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

is used in probability, statistics, and engineering.

   (a) Show that $\int_a^b e^{-t^2}\, dt = \frac{1}{2}\sqrt{\pi}[\operatorname{erf}(b) - \operatorname{erf}(a)]$.

   (b) Show that the function $y = e^{x^2} \operatorname{erf}(x)$ satisfies the differential equation $y' = 2xy + \frac{2}{\sqrt{\pi}}$.

4. Prove that for all $x > 0$ and all positive integers n,

$$e^x > 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

where $n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$.

5. (**90' Calculus Exam**)

   (a) Use integration by parts to show that if f has an inverse with continuous first derivative, then

   $$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x(f^{-1})'(x)\,dx.$$

   (b) If $\int f(x)\,dx = F(x) + C$, express $\int f^{-1}(x)\,dx$ in terms of F.

   (c) Calculate $\int \tan^{-1} x\,dx$.

6. (a) Show that $\cos(x^2) \geq \cos x$ for $0 \leq x \leq 1$.

   (b) Deduce that $\int_0^{\frac{\pi}{6}} \cos(x^2)\,dx \geq \frac{1}{2}$.

7. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

   $$\int_0^{\pi} xf(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx.$$

8. (a) If f is continuous, prove that

   $$\int_0^{\frac{\pi}{2}} f(\cos x)\,dx = \int_0^{\frac{\pi}{2}} f(\sin x)\,dx.$$

   (b) Use part (a) to evaluate

   $$\int_0^{\frac{\pi}{2}} \cos^2 x\,dx \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin^2 x\,dx.$$

9. (**Mean value theorem for integrals**) Suppose f is continuous on $[a, b]$, show that there exists $c \in (a, b)$ such that

   $$\int_a^b f(x)\,dx = f(c)(b - a).$$

10. Let f be a function such that f' is continuous on $[a, b]$. Show that

    $$\int_a^b f(t)f'(t)\,dt = \frac{1}{2}\left[(f(b))^2 - (f(a))^2\right].$$